9.3 THE SIMPLEX METHOD: MAXIMIZATION

For linear programming problems involving two variables, the graphical solution method introduced in Section 9.2 is convenient. However, for problems involving more than two variables or problems involving a large number of constraints, it is better to use solution methods that are adaptable to computers. One such method is called the **simplex method,** developed by George Dantzig in 1946. It provides us with a systematic way of examining the vertices of the feasible region to determine the optimal value of the objective function. We introduce this method with an example.

Suppose we want to find the maximum value of $z = 4x_1 + 6x_2$, where $x_1 \geq 0$ and $x_2 \geq 0$, subject to the following constraints.

$$\begin{aligned}
-x_1 + x_2 &\leq 11 \\
x_1 + x_2 &\leq 27 \\
2x_1 + 5x_2 &\leq 90
\end{aligned}$$

Since the left-hand side of each *inequality* is less than or equal to the right-hand side, there must exist nonnegative numbers s_1, s_2 and s_3 that can be added to the left side of each equation to produce the following system of linear *equations*.

$$\begin{aligned}
-x_1 + x_2 + s_1 \quad\quad\quad &= 11 \\
x_1 + x_2 \quad + s_2 \quad\quad &= 27 \\
2x_1 + 5x_2 \quad\quad + s_3 &= 90
\end{aligned}$$

The numbers s_1, s_2 and s_3 are called **slack variables** because they take up the "slack" in each inequality.

Standard Form of a Linear Programming Problem

A linear programming problem is in **standard form** if it seeks to *maximize* the objective function $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ subject to the constraints

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m
\end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. After adding slack variables, the corresponding system of **constraint equations** is

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 \quad\quad\quad\quad &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \quad + s_2 \quad\quad &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \quad\quad\quad + s_m &= b_m
\end{aligned}$$

where $s_i \geq 0$.

REMARK: Note that for a linear programming problem in standard form, the objective function is to be maximized, not minimized. (Minimization problems will be discussed in Sections 9.4 and 9.5.)

A **basic solution** of a linear programming problem in standard form is a solution $(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m)$ of the constraint equations in which *at most m* variables are nonzero—the variables that are nonzero are called **basic variables.** A basic solution for which all variables are nonnegative is called a **basic feasible solution.**

The Simplex Tableau

The simplex method is carried out by performing elementary row operations on a matrix that we call the **simplex tableau.** This tableau consists of the augmented matrix corresponding to the constraint equations together with the coefficients of the objective function written in the form

$$-c_1x_1 - c_2x_2 - \cdots - c_rx_n + (0)s_1 + (0)s_2 + \cdots + (0)\, s_m + z = 0.$$

In the tableau, it is customary to omit the coefficient of z. For instance, the simplex tableau for the linear programming problem

$$z = 4x_1 + 6x_2 \qquad \qquad \text{Objective function}$$

$$\left. \begin{array}{l} -x_1 + x_2 + s_1 \qquad\qquad = 11 \\ x_1 + x_2 \qquad + s_2 \qquad = 27 \\ 2x_1 + 5x_2 \qquad\qquad + s_3 = 90 \end{array} \right\} \qquad \text{Constraints}$$

is as follows.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	

↑
Current z–value

For this **initial simplex tableau,** the **basic variables** are $s_1, s_2,$ and $s_3,$ and the **nonbasic variables** (which have a value of zero) are x_1 and x_2. Hence, from the two columns that are farthest to the right, we see that the current solution is

$$x_1 = 0, \quad x_2 = 0, \quad s_1 = 11, \quad s_2 = 27, \quad \text{and} \quad s_3 = 90.$$

This solution is a basic feasible solution and is often written as

$$(x_1, x_2, s_1, s_2, s_3) = (0, 0, 11, 27, 90).$$

The entry in the lower–right corner of the simplex tableau is the current value of z. Note that the bottom–row entries under x_1 and x_2 are the negatives of the coefficients of x_1 and x_2 in the objective function

$$z = 4x_1 + 6x_2.$$

To perform an **optimality check** for a solution represented by a simplex tableau, we look at the entries in the bottom row of the tableau. If any of these entries are negative (as above), then the current solution is *not* optimal.

Pivoting

Once we have set up the initial simplex tableau for a linear programming problem, the simplex method consists of checking for optimality and then, if the current solution is not optimal, improving the current solution. (An improved solution is one that has a larger z-value than the current solution.) To improve the current solution, we bring a new basic variable into the solution—we call this variable the **entering variable.** This implies that one of the current basic variables must leave, otherwise we would have too many variables for a basic solution—we call this variable the **departing variable.** We choose the entering and departing variables as follows.

1. The **entering variable** corresponds to the smallest (the most negative) entry in the bottom row of the tableau.
2. The **departing variable** corresponds to the smallest nonnegative ratio of b_i/a_{ij}, in the column determined by the entering variable.
3. The entry in the simplex tableau in the entering variable's column and the departing variable's row is called the **pivot.**

Finally, to form the improved solution, we apply Gauss-Jordan elimination to the column that contains the pivot, as illustrated in the following example. (This process is called **pivoting.**)

EXAMPLE 1   *Pivoting to Find an Improved Solution*

Use the simplex method to find an improved solution for the linear programming problem represented by the following tableau.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	

The objective function for this problem is $z = 4x_1 + 6x_2$.

Solution   Note that the current solution $(x_1 = 0, x_2 = 0, s_1 = 11, s_2 = 27, s_3 = 90)$ corresponds to a z–value of 0. To improve this solution, we determine that x_2 is the entering variable, because -6 is the smallest entry in the bottom row.

| | | | | | | Basic |
x_1	x_2	s_1	s_2	s_3	b	Variables
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	

       ↑
    Entering

To see *why* we choose x_2 as the entering variable, remember that $z = 4x_1 + 6x_2$. Hence, it appears that a unit change in x_2 produces a change of 6 in z, whereas a unit change in x_1 produces a change of only 4 in z.

To find the departing variable, we locate the b_i's that have corresponding positive elements in the entering variables column and form the following ratios.

$$\frac{11}{1} = 11, \quad \frac{27}{1} = 27, \quad \frac{90}{5} = 18$$

Here the smallest positive ratio is 11, so we choose s_1 as the departing variable.

| | | | | | | Basic | |
x_1	x_2	s_1	s_2	s_3	b	Variables	
-1	①	1	0	0	11	s_1	← *Departing*
1	1	0	1	0	27	s_2	
2	5	0	0	1	90	s_3	
-4	-6	0	0	0	0		

       ↑
    Entering

Note that the pivot is the entry in the first row and second column. Now, we use Gauss-Jordan elimination to obtain the following improved solution.

Before Pivoting
$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 1 & 1 & 0 & 1 & 0 & 27 \\ 2 & 5 & 0 & 0 & 1 & 90 \\ -4 & -6 & 0 & 0 & 0 & 0 \end{bmatrix}$$

→

After Pivoting
$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$$

The new tableau now appears as follows.

| | | | | | | *Basic* |
x_1	x_2	s_1	s_2	s_3	b	*Variables*
-1	1	1	0	0	11	x_2
2	0	-1	1	0	16	s_2
7	0	-5	0	1	35	s_3
-10	0	6	0	0	66	

Note that x_2 has replaced s_1 in the basis column and the improved solution

$$(x_1, x_2, s_1, s_2, s_3) = (0, 11, 0, 16, 35)$$

has a z-value of

$$z = 4x_1 + 6x_2 = 4(0) + 6(11) = 66.$$

In Example 1 the improved solution is not yet optimal since the bottom row still has a negative entry. Thus, we can apply another iteration of the simplex method to further improve our solution as follows. We choose x_1 as the entering variable. Moreover, the smallest nonnegative ratio of $11/(-1)$, $16/2 = 8$, and $35/7 = 5$ is 5, so s_3 is the departing variable. Gauss-Jordan elimination produces the following.

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & 16 \\ 0 & 0 & \frac{3}{7} & 1 & -\frac{2}{7} & 6 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ 0 & 0 & -\frac{8}{7} & 0 & \frac{10}{7} & 116 \end{bmatrix}$$

Thus, the new simplex tableau is as follows.

| | | | | | | *Basic* |
x_1	x_2	s_1	s_2	s_3	b	*Variables*
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116	

In this tableau, there is still a negative entry in the bottom row. Thus, we choose s_1 as the entering variable and s_2 as the departing variable, as shown in the following tableau.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\left(\frac{3}{7}\right)$	1	$-\frac{2}{7}$	6	s_2 ← Departing
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116	

↑
Entering

By performing one more iteration of the simplex method, we obtain the following tableau. (Try checking this.)

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	12	x_2
0	0	1	$\frac{7}{3}$	$-\frac{2}{3}$	14	s_1
1	0	0	$\frac{5}{3}$	$-\frac{1}{3}$	15	x_1
0	0	0	$\frac{8}{3}$	$\frac{2}{3}$	132	← *Maximum z-value*

In this tableau, there are no negative elements in the bottom row. We have therefore determined the optimal solution to be

$$(x_1, x_2, s_1, s_2, s_3) = (15, 12, 14, 0, 0)$$

with

$$z = 4x_1 + 6x_2 = 4(15) + 6(12) = 132.$$

REMARK: Ties may occur in choosing entering and/or departing variables. Should this happen, any choice among the tied variables may be made.

Because the linear programming problem in Example 1 involved only two decision variables, we could have used a graphical solution technique, as we did in Example 2, Section 9.2. Notice in Figure 9.18 that each iteration in the simplex method corresponds to moving from a given vertex to an adjacent vertex with an improved z-value.

Figure 9.18

$(0, 0)$	→	$(0, 11)$	→	$(5, 16)$	→	$(15, 12)$
$z = 0$		$z = 66$		$z = 116$		$z = 132$
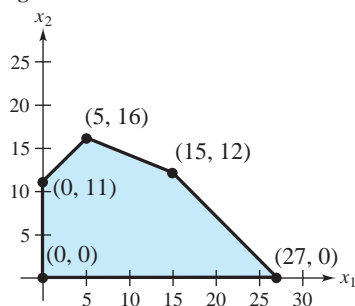
The Simplex Method

We summarize the steps involved in the simplex method as follows.

The Simplex Method (Standard Form)

To solve a linear programming problem in standard form, use the following steps.
1. Convert each inequality in the set of constraints to an equation by adding slack variables.
2. Create the initial simplex tableau.
3. Locate the most negative entry in the bottom row. The column for this entry is called the **entering column.** (If ties occur, any of the tied entries can be used to determine the entering column.)
4. Form the ratios of the entries in the "*b*-column" with their corresponding positive entries in the entering column. The **departing row** corresponds to the smallest non-negative ratio b_i/a_{ij}. (If all entries in the entering column are 0 or negative, then there is no maximum solution. For ties, choose either entry.) The entry in the departing row and the entering column is called the **pivot.**
5. Use elementary row operations so that the pivot is 1, and all other entries in the entering column are 0. This process is called **pivoting.**
6. If all entries in the bottom row are zero or positive, this is the final tableau. If not, go back to Step 3.
7. If you obtain a final tableau, then the linear programming problem has a maximum solution, which is given by the entry in the lower-right corner of the tableau.

Note that the basic feasible solution of an initial simplex tableau is

$$(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m) = (0, 0, \ldots, 0, b_1, b_2, \ldots, b_m).$$

This solution is basic because at most m variables are nonzero (namely the slack variables). It is feasible because each variable is nonnegative.

In the next two examples, we illustrate the use of the simplex method to solve a problem involving three decision variables.

EXAMPLE 2   **The Simplex Method with Three Decision Variables**

Use the simplex method to find the maximum value of

$$z = 2x_1 - x_2 + 2x_3 \qquad \text{Objective function}$$

subject to the constraints

$$2x_1 + x_2 \qquad \leq 10$$
$$x_1 + 2x_2 - 2x_3 \leq 20$$
$$x_2 + 2x_3 \leq 5$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

Solution   Using the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 10, 20, 5)$$

the initial simplex tableau for this problem is as follows. (Try checking these computations, and note the "tie" that occurs when choosing the first entering variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
2	1	0	1	0	0	10	s_1
1	2	−2	0	1	0	20	s_2
0	1	⟨2⟩	0	0	1	5	s_3 ← *Departing*
−2	1	−2	0	0	0	0	

↑
Entering

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
⟨2⟩	1	0	1	0	0	10	s_1 ← *Departing*
1	3	0	0	1	1	25	s_2
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3
−2	2	0	0	0	1	5	

↑
Entering

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	5	x_1
0	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	1	20	s_2
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3
0	3	0	1	0	1	15	

This implies that the optimal solution is

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (5, 0, \tfrac{5}{2}, 0, 20, 0)$$

and the maximum value of z is 15.

Occasionally, the constraints in a linear programming problem will include an equation. In such cases, we still add a "slack variable" called an **artificial variable** to form the initial simplex tableau. Technically, this new variable is not a slack variable (because there is no slack to be taken). Once you have determined an optimal solution in such a problem, you should check to see that any equations given in the original constraints are satisfied. Example 3 illustrates such a case.

EXAMPLE 3   *The Simplex Method with Three Decision Variables*

Use the simplex method to find the maximum value of

$$z = 3x_1 + 2x_2 + x_3 \qquad \textbf{Objective function}$$

subject to the constraints

$$4x_1 + x_2 + x_3 = 30$$
$$2x_1 + 3x_2 + x_3 \le 60$$
$$x_1 + 2x_2 + 3x_3 \le 40$$

where $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$.

Solution   Using the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 30, 60, 40)$$

the initial simplex tableau for this problem is as follows. (Note that s_1 is an artificial variable, rather than a slack variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
④	1	1	1	0	0	30	s_1	← *Departing*
2	3	1	0	1	0	60	s_2	
1	2	3	0	0	1	40	s_3	
−3	−2	−1	0	0	0	0		

↑
Entering

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{15}{2}$	x_1	
0	$\frac{5}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	45	s_2	← *Departing*
0	$\frac{7}{4}$	$\frac{11}{4}$	$-\frac{1}{4}$	0	1	$\frac{65}{2}$	s_3	
0	$-\frac{5}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	0	0	$\frac{45}{2}$		

↑
Entering

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	0	$\frac{1}{5}$	$\frac{3}{10}$	$-\frac{1}{10}$	0	3	x_1
0	1	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	0	18	x_2
0	0	$\frac{12}{5}$	$\frac{1}{10}$	$-\frac{7}{10}$	1	1	s_3
0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	45	

This implies that the optimal solution is

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (3, 18, 0, 0, 0, 1)$$

and the maximum value of z is 45. (This solution satisfies the equation given in the constraints because $4(3) + 1(18) + 1(0) = 30$.)

Applications

EXAMPLE 4   *A Business Application: Maximum Profit*

A manufacturer produces three types of plastic fixtures. The time required for molding, trimming, and packaging is given in Table 9.1. (Times are given in hours per dozen fixtures.)

TABLE 9.1

Process	Type A	Type B	Type C	Total time available
Molding	1	2	$\frac{3}{2}$	12,000
Trimming	$\frac{2}{3}$	$\frac{2}{3}$	1	4,600
Packaging	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	2,400
Profit	$11	$16	$15	—

How many dozen of each type of fixture should be produced to obtain a maximum profit?

Solution   Letting x_1, x_2, and x_3 represent the number of dozen units of Types A, B, and C, respectively, the objective function is given by

$$\text{Profit} = P = 11x_1 + 16x_2 + 15x_3.$$

Moreover, using the information in the table, we construct the following constraints.

$$x_1 + 2x_2 + \tfrac{3}{2}x_3 \le 12{,}000$$
$$\tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 + x_3 \le 4{,}600$$
$$\tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{2}x_3 \le 2{,}400$$

(We also assume that $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$.) Now, applying the simplex method with the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 12{,}000, 4{,}600, 2{,}400)$$

we obtain the following tableaus.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
1	⟨2⟩	$\frac{3}{2}$	1	0	0	12,000	s_1	← Departing
$\frac{2}{3}$	$\frac{2}{3}$	1	0	1	0	4,600	s_2	
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	0	0	1	2,400	s_3	
-11	-16	-15	0	0	0	0		

↑
Entering

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
$\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{1}{2}$	0	0	6,000	x_2
$\frac{1}{3}$	0	$\frac{1}{2}$	$-\frac{1}{3}$	1	0	600	s_2
$\left(\frac{1}{3}\right)$	0	$\frac{1}{4}$	$-\frac{1}{6}$	0	1	400	s_3 ← Departing
-3	0	-3	8	0	0	96,000	

↑
Entering

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
0	1	$\frac{3}{8}$	$\frac{3}{4}$	0	$-\frac{3}{2}$	5,400	x_2
0	0	$\left(\frac{1}{4}\right)$	$-\frac{1}{6}$	1	-1	200	s_2 ← Departing
1	0	$\frac{3}{4}$	$-\frac{1}{2}$	0	3	1,200	x_1
0	0	$-\frac{3}{4}$	$\frac{13}{2}$	0	9	99,600	

↑
Entering

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
0	1	0	1	$-\frac{3}{2}$	0	5,100	x_2
0	0	1	$-\frac{2}{3}$	4	-4	800	x_3
1	0	0	0	-3	6	600	x_1
0	0	0	6	3	6	100,200	

From this final simplex tableau, we see that the maximum profit is $100,200, and this is obtained by the following production levels.

Type A:	600 dozen units
Type B:	5,100 dozen units
Type C:	800 dozen units

REMARK: In Example 4, note that the second simplex tableau contains a "tie" for the minimum entry in the bottom row. (Both the first and third entries in the bottom row are -3.) Although we chose the first column to represent the departing variable, we could have chosen the third column. Try reworking the problem with this choice to see that you obtain the same solution.

EXAMPLE 5　*A Business Application: Media Selection*

The advertising alternatives for a company include television, radio, and newspaper advertisements. The costs and estimates for audience coverage are given in Table 9.2

TABLE 9.2

	Television	Newspaper	Radio
Cost per advertisement	$ 2,000	$ 600	$ 300
Audience per advertisement	100,000	40,000	18,000

The local newspaper limits the number of weekly advertisements from a single company to ten. Moreover, in order to balance the advertising among the three types of media, no more than half of the total number of advertisements should occur on the radio, and at least 10% should occur on television. The weekly advertising budget is $18,200. How many advertisements should be run in each of the three types of media to maximize the total audience?

Solution   To begin, we let x_1, x_2, and x_3 represent the number of advertisements in television, newspaper, and radio, respectively. The objective function (to be maximized) is therefore

$$z = 100{,}000x_1 + 40{,}000x_2 + 18{,}000x_3 \qquad \textbf{Objective function}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. The constraints for this problem are as follows.

$$2000x_1 + 600x_2 + 300x_3 \leq 18{,}200$$
$$x_2 \leq 10$$
$$x_3 \leq 0.5(x_1 + x_2 + x_3)$$
$$x_1 \geq 0.1(x_1 + x_2 + x_3)$$

A more manageable form of this system of constraints is as follows.

$$\left.\begin{array}{r} 20x_1 + 6x_2 + 3x_3 \leq 182 \\ x_2 \leq 10 \\ -x_1 - x_2 + x_3 \leq 0 \\ -9x_1 + x_2 + x_3 \leq 0 \end{array}\right\} \qquad \textbf{Constraints}$$

Thus, the initial simplex tableau is as follows.

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	Basic Variables	
20	6	3	1	0	0	0	182	s_1	← Departing
0	1	0	0	1	0	0	10	s_2	
−1	−1	1	0	0	1	0	0	s_3	
−9	1	1	0	0	0	1	0	s_4	
−100,000	−40,000	−18,000	0	0	0	0	0		

↑
Entering

Now, to this initial tableau, we apply the simplex method as follows.

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	Basic Variables
1	$\frac{3}{10}$	$\frac{3}{20}$	$\frac{1}{20}$	0	0	0	$\frac{91}{10}$	x_1
0	(1)	0	0	1	0	0	10	$s_2 \leftarrow$ *Departing*
0	$-\frac{7}{10}$	$\frac{23}{20}$	$\frac{1}{20}$	0	1	0	$\frac{91}{10}$	s_3
0	$\frac{37}{10}$	$\frac{47}{20}$	$\frac{9}{20}$	0	0	1	$\frac{819}{10}$	s_4
0	$-10{,}000$	$-3{,}000$	$5{,}000$	0	0	0	$910{,}000$	

$\uparrow$ *Entering* (under x_2)

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	Basic Variables
1	0	$\frac{3}{20}$	$\frac{1}{20}$	$-\frac{3}{10}$	0	0	$\frac{61}{10}$	x_1
0	1	0	0	1	0	0	10	x_2
0	0	$\left(\frac{23}{20}\right)$	$\frac{1}{20}$	$\frac{7}{10}$	1	0	$\frac{161}{10}$	$s_3 \leftarrow$ *Departing*
0	0	$\frac{47}{20}$	$\frac{9}{20}$	$-\frac{37}{10}$	0	1	$\frac{449}{10}$	s_4
0	0	$-3{,}000$	$5{,}000$	$10{,}000$	0	0	$1{,}010{,}000$	

$\uparrow$ *Entering* (under x_3)

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	Basic Variables
1	0	0	$\frac{1}{23}$	$-\frac{9}{23}$	$-\frac{3}{23}$	0	4	x_1
0	1	0	0	1	0	0	10	x_2
0	0	1	$\frac{1}{23}$	$\frac{14}{23}$	$\frac{20}{23}$	0	14	x_3
0	0	0	$\frac{8}{23}$	$-\frac{118}{23}$	$-\frac{47}{23}$	1	12	s_4
0	0	0	$\frac{118{,}000}{23}$	$\frac{272{,}000}{23}$	$\frac{60{,}000}{23}$	0	$1{,}052{,}000$	

From this tableau, we see that the maximum weekly audience for an advertising budget of $18,200 is

$$z = 1{,}052{,}000 \qquad \textbf{Maximum weekly audience}$$

and this occurs when $x_1 = 4$, $x_2 = 10$, and $x_3 = 14$. We sum up the results here.

Media	Number of Advertisements	Cost	Audience
Television	4	$ 8,000	400,000
Newspaper	10	$ 6,000	400,000
Radio	14	$ 4,200	252,000
Total	28	$18,200	1,052,000

SECTION 9.3 ☐ EXERCISES

In Exercises 1–4, write the simplex tableau for the given linear programming problem. You do not need to solve the problem. (In each case the objective function is to be maximized.)

1. Objective function:
$$z = x_1 + 2x_2$$
Constraints:
$$2x_1 + x_2 \le 8$$
$$x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

2. Objective function:
$$z = x_1 + 3x_2$$
Constraints:
$$x_1 + x_2 \le 4$$
$$x_1 - x_2 \le 1$$
$$x_1, x_2 \ge 0$$

3. Objective function:
$$z = 2x_1 + 3x_2 + 4x_3$$
Constraints:
$$x_1 + 2x_2 \quad \le 12$$
$$x_1 \quad + x_3 \le 8$$
$$x_1, x_2, x_3 \ge 0$$

4. Objective function:
$$z = 6x_1 - 9x_2$$
Constraints:
$$2x_1 - 3x_2 \le 6$$
$$x_1 + x_2 \le 20$$
$$x_1, x_2 \ge 0$$

In Exercises 5–8, explain why the linear programming problem is *not* in standard form as given.

5. (Minimize)
Objective function:
$$z = x_1 + x_2$$
Constraints:
$$x_1 + 2x_2 \le 4$$
$$x_1, x_2 \ge 0$$

6. (Maximize)
Objective function:
$$z = x_1 + x_2$$
Constraints:
$$x_1 + 2x_2 \le 6$$
$$2x_1 - x_2 \le -1$$
$$x_1, x_2 \ge 0$$

7. (Maximize)
Objective function:
$$z = x_1 + x_2$$
Constraints:
$$x_1 + x_2 + 3x_3 \le 5$$
$$2x_1 \quad - 2x_3 \ge 1$$
$$x_2 + x_3 \le 0$$
$$x_1, x_2, x_3 \ge 0$$

8. (Maximize)
Objective function:
$$z = x_1 + x_2$$
Constraints:
$$x_1 + x_2 \ge 4$$
$$2x_1 + x_2 \ge 6$$
$$x_1, x_2 \ge 0$$

In Exercises 9–20, use the simplex method to solve the given linear programming problem. (In each case the objective function is to be maximized.)

9. Objective function:
$$z = x_1 + 2x_2$$
Constraints:
$$x_1 + 4x_2 \le 8$$
$$x_1 + x_2 \le 12$$
$$x_1, x_2 \ge 0$$

10. Objective function:
$$z = x_1 + x_2$$
Constraints:
$$x_1 + 2x_2 \le 6$$
$$3x_1 + 2x_2 \le 12$$
$$x_1, x_2 \ge 0$$

11. Objective function:
$$z = 5x_1 + 2x_2 + 8x_3$$
Constraints:
$$2x_1 - 4x_2 + x_3 \le 42$$
$$2x_1 + 3x_2 - x_3 \le 42$$
$$6x_1 - x_2 + 3x_3 \le 42$$
$$x_1, x_2, x_3 \ge 0$$

12. Objective function:
$$z = x_1 - x_2 + 2x_3$$
Constraints:
$$2x_1 + 2x_2 \le 8$$
$$x_3 \le 5$$
$$x_1, x_2, x_3 \ge 0$$

13. Objective function:
$$z = 4x_1 + 5x_2$$
Constraints:
$$x_1 + x_2 \le 10$$
$$3x_1 + 7x_2 \le 42$$
$$x_1, x_2 \ge 0$$

14. Objective function:
$$z = x_1 + 2x_2$$
Constraints:
$$x_1 + 3x_2 \le 15$$
$$2x_1 - x_2 \le 12$$
$$x_1, x_2 \ge 0$$

15. Objective function:
$$z = 3x_1 + 4x_2 + x_3 + 7x_4$$
Constraints:
$$8x_1 + 3x_2 + 4x_3 + x_4 \le 7$$
$$2x_1 + 6x_2 + x_3 + 5x_4 \le 3$$
$$x_1 + 4x_2 + 5x_3 + 2x_4 \le 8$$
$$x_1, x_2, x_3, x_4 \ge 0$$

16. Objective function:
$$z = x_1$$
Constraints:
$$3x_1 + 2x_2 \le 60$$
$$x_1 + 2x_2 \le 28$$
$$x_1 + 4x_2 \le 48$$
$$x_1, x_2 \ge 0$$

17. Objective function:
$$z = x_1 - x_2 + x_3$$
Constraints:
$$2x_1 + x_2 - 3x_3 \le 40$$
$$x_1 \quad + x_3 \le 25$$
$$2x_2 + 3x_3 \le 32$$
$$x_1, x_2, x_3 \ge 0$$

18. Objective function:
$$z = 2x_1 + x_2 + 3x_3$$
Constraints:
$$x_1 + x_2 + x_3 \le 59$$
$$2x_1 \quad + 3x_3 \le 75$$
$$x_2 + 6x_3 \le 54$$
$$x_1, x_2, x_3 \ge 0$$

19. Objective function:
$$z = x_1 + 2x_2 - x_4$$
Constraints:
$$x_1 + 2x_2 + 3x_3 \quad \le 24$$
$$3x_2 + 7x_3 + x_4 \le 42$$
$$x_1, x_2, x_3, x_4 \ge 0$$

20. Objective function:
$$z = x_1 + 2x_2 + x_3 - x_4$$
Constraints:
$$x_1 + x_2 + 3x_3 + 4x_4 \le 60$$
$$x_2 + 2x_3 + 5x_4 \le 50$$
$$2x_1 + 3x_2 \quad + 6x_4 \le 72$$
$$x_1, x_2, x_3, x_4 \ge 0$$

21. A merchant plans to sell two models of home computers at costs of $250 and $400, respectively. The $250 model yields a profit of $45 and the $400 model yields a profit of $50. The merchant estimates that the total monthly demand will not exceed 250 units. Find the number of units of each model that should be stocked in order to maximize profit. Assume that the merchant does not want to invest more than $70,000 in computer inventory. (See Exercise 21 in Section 9.2.)

22. A fruit grower has 150 acres of land available to raise two crops, A and B. It takes one day to trim an acre of crop A and two days to trim an acre of crop B, and there are 240 days per year available for trimming. It takes 0.3 day to pick an acre of crop A and 0.1 day to pick an acre of crop B, and there are 30 days per year available for picking. Find the number of acres of each fruit that should be planted to maximize profit, assuming that the profit is $140 per acre for crop A and $235 per acre for B. (See Exercise 22 in Section 9.2.)

23. A grower has 50 acres of land for which she plans to raise three crops. It costs $200 to produce an acre of carrots and the profit is $60 per acre. It costs $80 to produce an acre of celery and the profit is $20 per acre. Finally, it costs $140 to produce an acre of lettuce and the profit is $30 per acre. Use the simplex method to find the number of acres of each crop she should plant in order to maximize her profit. Assume that her cost cannot exceed $10,000.

24. A fruit juice company makes two special drinks by blending apple and pineapple juices. The first drink uses 30% apple juice and 70% pineapple, while the second drink uses 60% apple and 40% pineapple. There are 1000 liters of apple and 1500 liters of pineapple juice available. If the profit for the first drink is $0.60 per liter and that for the second drink is $0.50, use the simplex method to find the number of liters of each drink that should be produced in order to maximize the profit.

25. A manufacturer produces three models of bicycles. The time (in hours) required for assembling, painting, and packaging each model is as follows.

	Model A	Model B	Model C
Assembling	2	2.5	3
Painting	1.5	2	1
Packaging	1	0.75	1.25

The total time available for assembling, painting, and packaging is 4006 hours, 2495 hours and 1500 hours, respectively. The profit per unit for each model is $45 (Model A), $50 (Model B), and $55 (Model C). How many of each type should be produced to obtain a maximum profit?

26. Suppose in Exercise 25 the total time available for assembling, painting, and packaging is 4000 hours, 2500 hours, and 1500 hours, respectively, and that the profit per unit is $48 (Model A), $50 (Model B), and $52 (Model C). How many of each type should be produced to obtain a maximum profit?

27. A company has budgeted a maximum of $600,000 for advertising a certain product nationally. Each minute of television time costs $60,000 and each one-page newspaper ad costs $15,000. Each television ad is expected to be viewed by 15 million viewers, and each newspaper ad is expected to be seen by 3 million readers. The company's market research department advises the company to use at most 90% of the advertising budget on television ads. How should the advertising budget be allocated to maximize the total audience?

28. Rework Exercise 27 assuming that each one-page newspaper ad costs $30,000.

29. An investor has up to $250,000 to invest in three types of investments. Type A pays 8% annually and has a risk factor of 0. Type B pays 10% annually and has a risk factor of 0.06. Type C pays 14% annually and has a risk factor of 0.10. To have a well-balanced portfolio, the investor imposes the following conditions. The average risk factor should be no greater than 0.05. Moreover, at least one-fourth of the total portfolio is to be allocated to Type A investments and at least one-fourth of the portfolio is to be allocated to Type B investments. How much should be allocated to each type of investment to obtain a maximum return?

30. An investor has up to $450,000 to invest in three types of investments. Type A pays 6% annually and has a risk factor of 0. Type B pays 10% annually and has a risk factor of 0.06. Type C pays 12% annually and has a risk factor of 0.08. To have a well-balanced portfolio, the investor imposes the following conditions. The average risk factor should be no greater than 0.05. Moreover, at least one-half of the total portfolio is to be allocated to Type A investments and at least one-fourth of the portfolio is to be allocated to Type B investments. How much should be allocated to each type of investment to obtain a maximum return?

31. An accounting firm has 900 hours of staff time and 100 hours of reviewing time available each week. The firm charges $2000 for an audit and $300 for a tax return. Each audit requires 100 hours of staff time and 10 hours of review time, and each tax return requires 12.5 hours of staff time and 2.5 hours of review time. What number of audits and tax returns will bring in a maximum revenue?

32. The accounting firm in Exercise 31 raises its charge for an audit to $2500. What number of audits and tax returns will bring in a maximum revenue?

In the simplex method, it may happen that in selecting the departing variable all the calculated ratios are negative. This indicates an *unbounded solution*. Demonstrate this in Exercises 33 and 34.

33. (Maximize)

Objective function:

$z = x_1 + 2x_2$

Constraints:

$$x_1 - 3x_2 \leq 1$$
$$-x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

34. (Maximize)

Objective function:

$z = x_1 + 3x_2$

Constraints:

$$-x_1 + x_2 \leq 20$$
$$-2x_1 + x_2 \leq 50$$
$$x_1, x_2 \geq 0$$

If the simplex method terminates and one or more variables *not in the final basis* have bottom-row entries of zero, bringing these variables into the basis will determine other optimal solutions. Demonstrate this in Exercises 35 and 36.

35. (Maximize)

Objective function:

$z = 2.5x_1 + x_2$

Constraints:

$$3x_1 + 5x_2 \leq 15$$
$$5x_1 + 2x_2 \leq 10$$
$$x_1, x_2 \geq 0$$

36. (Maximize)

Objective function:

$z = x_1 + \frac{1}{2}x_2$

Constraints:

$$2x_1 + x_2 \leq 20$$
$$x_1 + 3x_2 \leq 35$$
$$x_1, x_2 \geq 0$$

C 37. Use a computer to maximize the objective function

$z = 2x_1 + 7x_2 + 6x_3 + 4x_4$

subject to the constraints

$$x_1 + x_2 + 0.83x_3 + 0.5x_4 \leq 65$$
$$1.2x_1 + x_2 + x_3 + 1.2x_4 \leq 96$$
$$0.5x_1 + 0.7x_2 + 1.2x_3 + 0.4x_4 \leq 80$$

where $x_1, x_2, x_3, x_4 \geq 0$.

C 38. Use a computer to maximize the objective function

$z = 1.2x_1 + x_2 + x_3 + x_4$

subject to the same set of constraints given in Exercise 37.

9.4 THE SIMPLEX METHOD: MINIMIZATION

In Section 9.3, we applied the simplex method only to linear programming problems in standard form where the objective function was to be *maximized*. In this section, we extend this procedure to linear programming problems in which the objective function is to be *minimized*.

A minimization problem is in **standard form** if the objective function $w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is to be minimized, subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m$$

where $x_i \geq 0$ and $b_i \geq 0$. The basic procedure used to solve such a problem is to convert it to a *maximization problem* in standard form, and then apply the simplex method as discussed in Section 9.3.

In Example 5 in Section 9.2, we used geometric methods to solve the following minimization problem.