Chapter 6 Linear Programming: The Simplex Method

We will now consider LP (Linear Programming) problems that involve more than 2 decision variables. We will learn an algorithm called the simplex method which will allow us to solve these kind of problems.

Maximization Problem in Standard Form

We start with defining the standard form of a linear programming problem which will make further discussion easier.

Definition.
A linear programming problem is said to be a **standard maximization problem in standard form** if its mathematical model is of the following form:

$$\begin{aligned} \text{Maximize} \quad & P = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\ & \ldots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

where $x_1, x_2, \ldots, x_n$ are decision variables, $c_1, \ldots, c_n$, $a_{11}, \ldots, a_{mn}$ are any real numbers, and $b_1, \ldots, b_m \geq 0$ are nonnegative real numbers.

Note: Any linear programming problem (in the form we defined earlier) can be converted intothe standard maximization problem in standard form.

Initial System and Slack Variables

Roughly speaking, the idea of the simplex method is to represent an LP problem as a system of linear equations, and then a certain solution (possessing some properties we will define later) of the obtained system would be an optimal solution of the initial LP problem (if any exists). The simplex method defines an efficient algorithm of finding this specific solution of the system of linear equations.

Therefore, we need to start with converting given LP problem into a system of linear equations. First, we convert problem constraints into equations with the help of **slack variables**.

Consider the following maximization problem in the standard form:

$$\begin{aligned} \text{Maximize} \quad & P = 5x_1 + 4x_2 \\ \text{subject to} \quad & 4x_1 + 2x_2 \leq 32 \\ & 2x_1 + 3x_2 \leq 24 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{1}$$

The variables s_1 and s_2 are called **slack variables** because each makes up the difference (takes up the slack) between the left and right sides of an inequality. **For each problem constraint of the original problem we introduce a single slack variable**.

Therefore, we get

$$\begin{aligned} 4x_1 + 2x_2 + s_1 &= 32 \\ 2x_1 + 3x_2 + s_2 &= 24 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned} \tag{2}$$

Note that each solution of (2) corresponds to a point in the feasible region of (1). Also note that the **slack variables should be non-negative** as well. If slack variable is negative, then the right-hand side of corresponding problem constrain should be larger than the left-hand, i.e., this constraint would be violated.

Now, we also add the objective function to (2) treating P as yet another variable.

$$P = 5x_1 + 4x_2 \implies$$

Therefore, we get

$$\begin{array}{rcl} 4x_1 + 2x_2 + s_1 \qquad\qquad & = & 32 \\ 2x_1 + 3x_2 \qquad + s_2 \qquad & = & 24 \\ -5x_1 - 4x_2 \qquad\qquad + P & = & 0 \\ x_1, x_2, s_1, s_2 & \geq & 0 \end{array}$$

The above system is called the **initial system**. Again, every solution of the initial system (taking into account nonnegative constrains) corresponds to some point in the feasible region of the original LP (**and vice versa!**), and in addition the initial system incorporates information about the objective function of the original LP. Therefore, **one of the solution of the initial system should be an optimal solution of the original LP** (if any exists). Clearly, the **initial system has infinitely many solutions**, so the key question is **which one of these solutions is an optimal solution of the LP**?

Basic Solutions and Basic Feasible Solutions

We now define two important types of solutions of the initial systems that we should focus our attention on in order to identify the optimal solution of the LP.

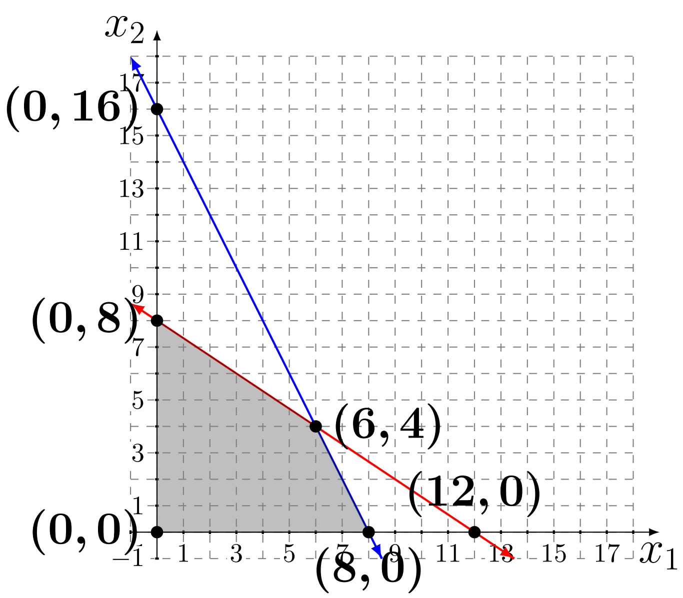
Definition (Basic Solution)

Given an LP with n decision variables and m constraints, a **basic solution** of the corresponding initial system is a solution of the initial systems (not taking into account nonnegative constraints) in which n of the variables $x_1, \ldots, x_n, s_1, \ldots, s_m$ are equal to zero.
Note: the list of variables $x_1, \ldots, x_n, s_1, \ldots, s_m$, n of which should be zero, does not contain P.

Definition (Basic Feasible Solution)

If a basic solution of the initial system corresponds to a certain point in the feasible region of the original LP, then it is called a **basic feasible solution**.

The feasible region of (1) looks like



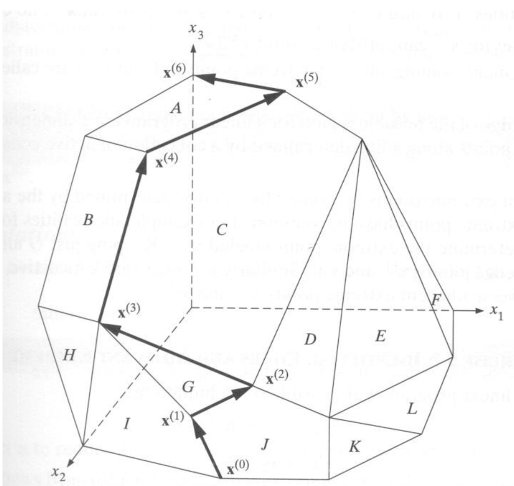
In 2-dimensional case (2 decision variables), the set of basic solutions is the of pairwise intersections of boundary lines of all problem constraints. In turn, the set of basic feasible solutions is the set of the corner points. Indeed,

Theorem 1 (Fundamental Theorem of Linear Programming: Another Version)

If the optimal value of the objective function in a linear programming problem exists, then that value must occur at one or more of the basic feasible solutions of the initial system.

So, by checking all basic solutions for feasibility and optimality we can solve any LP. In our example, this is quite easy because there are 6 basic solutions and just 4 of them are feasible. However, in a lot of real-world LP problems the number of variables and the number of constraints are much higher. For example, if a problem has $n = 30$ decision variables and $m = 35$ problem constraints, the number of possible basic solution becomes approximately $\mathbf{3 \times 10^{18}}$. It will take about **15 years** for an average modern personal computer to check all these solutions for feasibility and optimality.

The simplex method describes a "smart" way to find much smaller subset of basic solutions which would be sufficient to check in order to identify the optimal solution. Staring from some basic feasible solution called **initial basic feasible solution**, the simplex method moves along the edges of the polyhedron (vertices of which are basic feasible solutions) in **the direction of increase of the objective function** until it reaches the optimal solution.



Simplex Tableau

The simplex method utilizes matrix representation of the initial system while performing search for the optimal solution. This matrix representation is called **simplex tableau** and it is actually the augmented matrix of the initial systems with some additional information.

Let's write down the augmented matrix of the initial system corresponding to the LP (1).

$$\left[\begin{array}{rrrrr|r} 4 & 2 & 1 & 0 & 0 & 32 \\ 2 & 3 & 0 & 1 & 0 & 24 \\ \hline -5 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$$

At each step of the simplex method a particular basic feasible solution is considered. Information about this solutions is also represented in the simplex tableau. Recall that we defined a basic feasible solution as a solution with n variables being zero. In this context, we have

Definition (Basic and Nonbasic Variables)
The variables of a basic solution that are assumed to be zero are called **nonbasic** variables. All the remaining variables are called **basic** variables.

So at each step, we need define the list of basic and nonbasic variables. Note that if n nonbasic variables are assigned the value 0, the corresponding values of the nonbasic $m + 1$ variables can be determined by solving the corresponding system of linear equations.

Question: How do we decide which variables are basic and which are not? In particular, how can we decide what variables would be basic and what would be nonbasic on the very first step of the simplex method (how do we choose the initial basic feasible solution)?

PROCEDURE Selecting Basic and Nonbasic Variables for the Simplex Method

Given a simplex tableau,

Step 1 *Numbers of variables.* Determine the number of basic variables and the number of nonbasic variables. These numbers do not change during the simplex process.

Step 2 *Selecting basic variables.* A variable can be selected as a basic variable only if it corresponds to a column in the tableau that has exactly one nonzero element (usually 1) and the nonzero element in the column is not in the same row as the nonzero element in the column of another basic variable. This procedure always selects P as a basic variable, since the P column never changes during the simplex process.

Step 3 *Selecting nonbasic variables.* After the basic variables are selected in step 2, the remaining variables are selected as the nonbasic variables. The tableau columns under the nonbasic variables usually contain more than one nonzero element.

Therefore, for our example we have

$$\left[\begin{array}{rrrrr|r} 4 & 2 & 1 & 0 & 0 & 32 \\ 2 & 3 & 0 & 1 & 0 & 24 \\ \hline -5 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$$

Pivot Operation

So far, we set up a **simplex tableau** and identified the **initial basic feasible solution** by determining basic and nonbasic variables. This is the first step of the simplex method.

At each further step the simplex methods **swaps one of the non-basic variables for one of the basic variables** (so it moves to another vertex of the polyhedron) in the way such that the value of the **objective function is improved** (becomes higher). If improvement of the objective function is not possible, then we got an optimal solution.

Since we do not choose ourselves which variables are basic but rather determine them by reading the simplex tableau, in order for such swap to happen the simplex tableau should be changed. This is done with the help of **pivot operations**. However, before doing this transformation we need to decide ourselves which nonbasic variable should become basic and vice versa.

Definition (Entering and Exiting Variables)

A *nonbasic* variable that is chosen to become a *basic* variable at a particular step of the simplex method is called **entering variable**.

A *basic* variable that is chosen to become a *nonbasic* variable at a particular step of the simplex method is called **exiting variable**.

Getting back to our example

$$\left[\begin{array}{ccccc|c} 4 & 2 & 1 & 0 & 0 & 32 \\ 2 & 3 & 0 & 1 & 0 & 24 \\ \hline -5 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$$

Let's first select the **entering** variable:

Definition (Pivot Column)
The column corresponding to the *entering* variable is called the **pivot column**.

Now, let's select the **exiting** variable:

Definition (Pivot Row and Pivot Element)

The row corresponding to the *exiting* variable is called the **pivot row**.

The element at the intersection of the *pivot column* and the *pivot row* is called the **pivot element**.

So, we have

$$\left[\begin{array}{rrrrr|r} 4 & 2 & 1 & 0 & 0 & 32 \\ 2 & 3 & 0 & 1 & 0 & 24 \\ \hline -5 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$$

PROCEDURE Selecting the Pivot Element

Step 1 Locate the most negative indicator in the bottom row of the tableau to the left of the P column (the negative number with the largest absolute value). The column containing this element is the *pivot column*. If there is a tie for the most negative indicator, choose either column.

Step 2 Divide each *positive* element in the pivot column above the dashed line into the corresponding element in the last column. The *pivot row* is the row corresponding to the smallest quotient obtained. If there is a tie for the smallest quotient, choose either row. If the pivot column above the dashed line has no positive elements, there is no solution, and we stop.

Step 3 The *pivot* (or *pivot element*) is the element at the intersection of the pivot column and pivot row.

Note: The pivot element is always positive and never appears in the bottom row.

Remember: The entering variable is at the top of the pivot column, and the exiting variable is at the left of the pivot row.

Now, when we know which variable is entering and which is exiting we need to perform row operations on the tableau so that the **pivot element is transformed into 1 and all other elements in the column into 0's**. This procedure for transforming a nonbasic variable into a basic variable is called a **pivot operation**, or **pivoting**, and is summarized below.

PROCEDURE Performing a Pivot Operation

A **pivot operation**, or **pivoting**, consists of performing row operations as follows:

Step 1 Multiply the pivot row by the reciprocal of the pivot element to transform the pivot element into a 1. (If the pivot element is already a 1, omit this step.)

Step 2 Add multiples of the pivot row to other rows in the tableau to transform all other nonzero elements in the pivot column into 0's.

Performing a pivot operation has the following effects:

1. The (entering) nonbasic variable becomes a basic variable.
2. The (exiting) basic variable becomes a nonbasic variable.
3. The value of the objective function is increased, or, in some cases, remains the same.

Also, note that

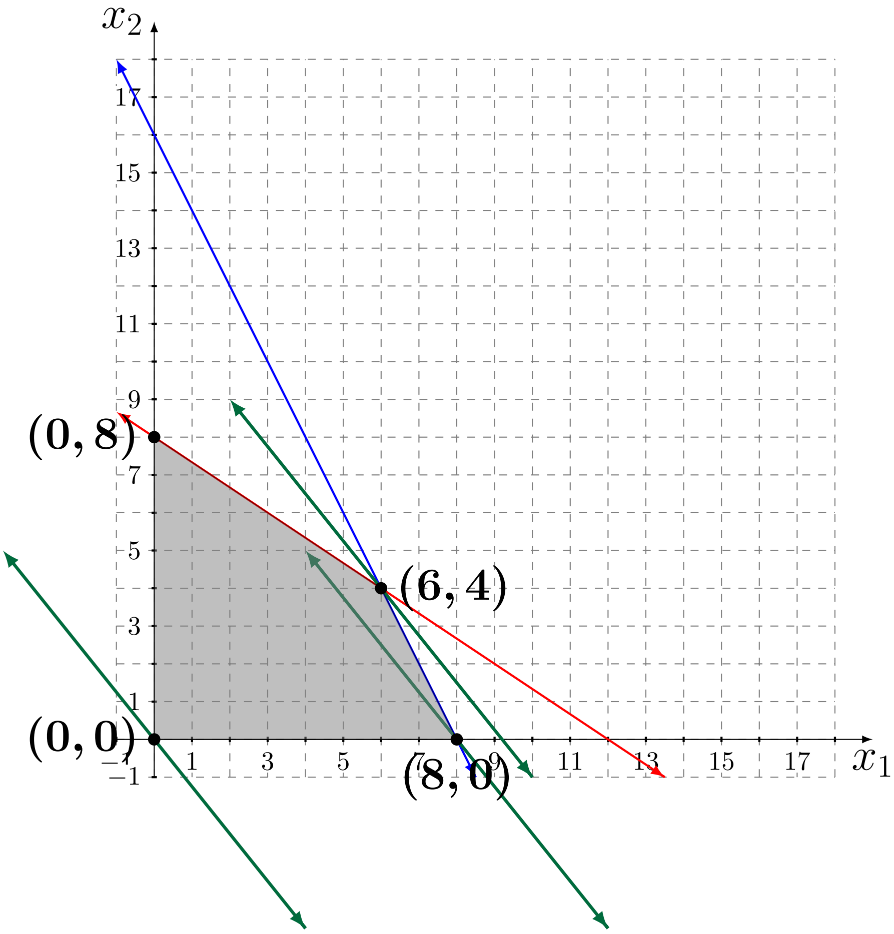
Do not interchange rows.

A pivot operation uses some of the same row operations as those used in GaussJordan elimination, but there is one essential difference. In a pivot operation, **you can never interchange two rows**.

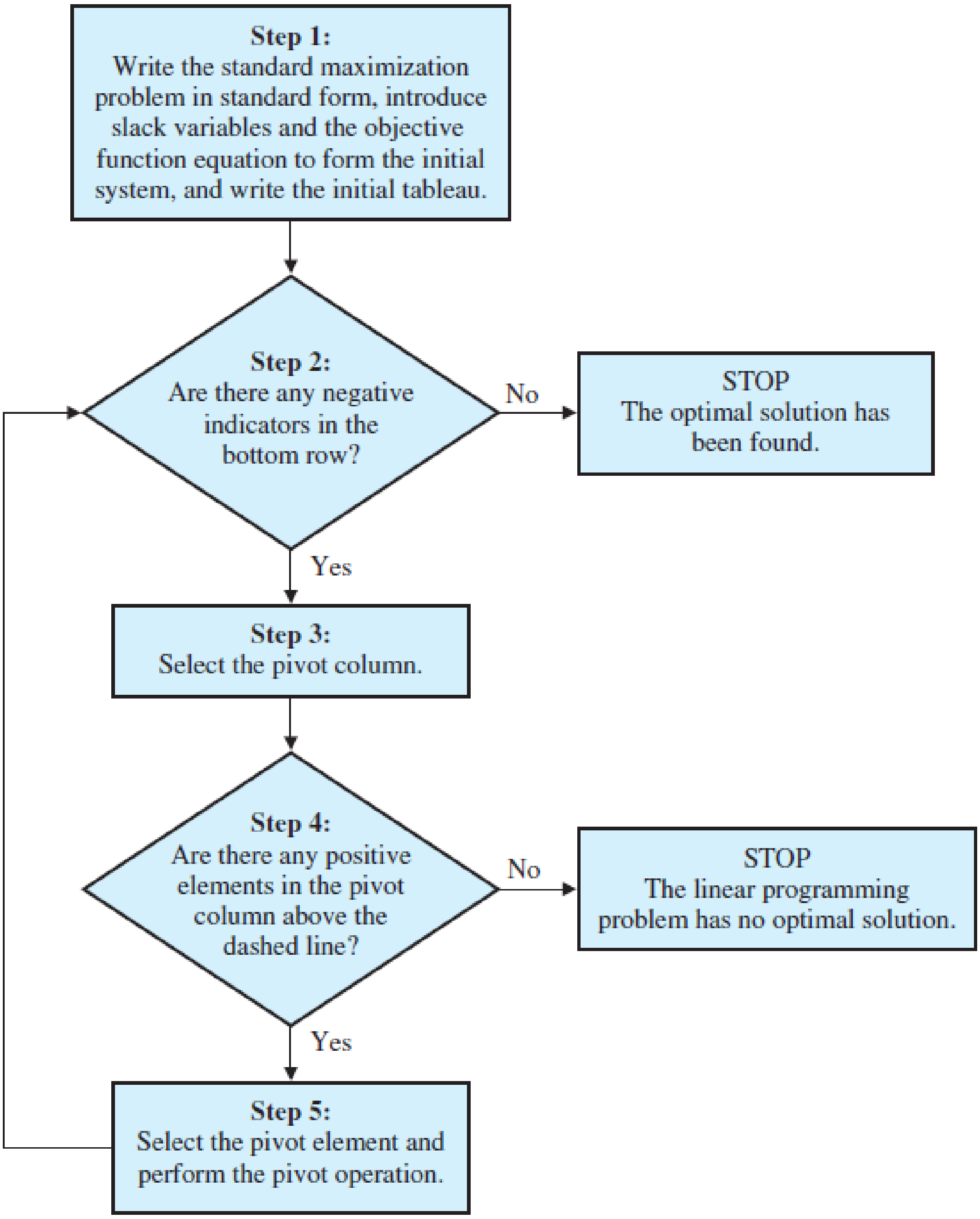
Getting back to our example

$$\left[\begin{array}{rrrrr|r} 4 & 2 & 1 & 0 & 0 & 32 \\ 2 & 3 & 0 & 1 & 0 & 24 \\ \hline -5 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$$

Interpreting the Simplex Process Geometrically



Summary



Step 1:
Write the standard maximization problem in standard form, introduce slack variables and the objective function equation to form the initial system, and write the initial tableau.
Step 2:
Are there any negative indicators in the bottom row?
No
STOP
The optimal solution has been found.
Yes
Step 3:
Select the pivot column.
Step 4:
Are there any positive elements in the pivot column above the dashed line?
No
STOP
The linear programming problem has no optimal solution.
Yes
Step 5:
Select the pivot element and perform the pivot operation.

More Examples

Example 1

$$
\begin{aligned}
\text{Maximize} \quad & P = 30x_1 + 40x_2 \\
\text{subject to} \quad & 2x_1 + x_2 \le 10 \\
& x_1 + x_2 \le 7 \\
& x_1 + 2x_2 \le 12 \\
& \qquad x_1, x_2 \ge 0
\end{aligned}
$$

Example 2

$$
\begin{aligned}
\text{Maximize} \quad & P = 6x_1 + 3x_2 \\
\text{subject to} \quad & -2x_1 + 3x_2 \leq 9 \\
& -x_1 + 3x_2 \leq 12 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Example 3

$$\begin{aligned}
\text{Maximize} \quad & P = 4x_1 + 3x_2 + 2x_3 \\
\text{subject to} \quad & 3x_1 + 2x_2 + 5x_3 \le 23 \\
& 2x_1 + x_2 + x_3 \le 8 \\
& x_1 + x_2 + 2x_3 \le 7 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}$$